

# 오스트팔리아 대학교의 한림 수학 AI 튜터 사례 연구

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- **약력**

- 2004: 박사 학위 취득
- 2009: 교수자격 취득 (독일 교육 시스템의 특성)
- 2005~2010: 소프트웨어 아키텍트로 활동
- 2010 이후: 오스트팔리아 대학교 재직



- **주요 연구 분야:** 그래프 이론

- **주요 강의 주제:** IT 보안, 데이터베이스, 운용 과학, 비즈니스 프로세스, 수학



## 사례 연구 배경

- 한림대학교 '수학 AI 튜터' 연구 프로젝트
- 초기 개발 단계
- 활발한 개발 및 주간 단위 업데이트
- 오스트팔리아 대학의 역할: 초기 사용 경험 추적 및 피드백 제공
- 목표 대상: 선형대수학을 수강하는 학부생



## 주요 연구 질문

- 풀이 과정에 대한 적절한 힌트 제공 여부
- 힌트의 상세 수준: 너무 많거나 적지 않은가?
  - 이전 힌트와 다음 힌트의 일관성
  - 계산 오류의 정확한 탐지 및 지적
- 독일과 한국 간의 문화적, 교육적 차이

# 수학 AI 튜터: 인터페이스 및 워크플로우



AI Math Tutor v1.5.0

English

Guide

Send Feedback

T. Sander

Linear Equations > Systems of Linear Equations

Problem 2

Compute the unique solution of the following linear system:

$$-\frac{2}{3}x_1 + 0x_2 - \frac{4}{3}x_3 + \frac{1}{6}x_4 = -1$$

$$\frac{6}{5}x_1 - \frac{2}{3}x_2 - \frac{5}{2}x_3 - \frac{2}{5}x_4 = \frac{13}{30}$$

$$\frac{6}{5}x_1 - 2x_2 + \frac{3}{2}x_3 + 0x_4 = \frac{103}{10}$$

$$-2x_1 + \frac{5}{2}x_2 - 1x_3 - \frac{4}{5}x_4 = \frac{-86}{15}$$

### Recognition Result

Solution:

$$\begin{bmatrix} 12 & 8 & 16 \\ 5 & 1 & 3 \end{bmatrix} \quad 8 \cdot \frac{-1}{12} = \frac{-8}{12} = -\frac{2}{3}$$

$$R_1 \leftarrow \frac{-1}{12}R_1$$

$$R_2 \leftarrow R_2 - 5R_1$$

Please write your solution steps (supports formulas, numbers, and English)

$$\begin{bmatrix} 12 & 8 & 16 \\ 5 & 1 & 3 \end{bmatrix}$$

$$R_1 \leftarrow \frac{-1}{12}R_1$$

$$R_2 \leftarrow R_2 - 5R_1$$

$$8 \cdot \frac{-1}{12} = \frac{-8}{12} = -\frac{2}{3}$$

system only defines  $x_1, x_2, x_3, x_4$ . Please recheck the variable indexing and express your solution in terms of  $x_1, x_2, x_3, x_4$ .

Your matrix doesn't represent the system: it has only 2 rows and 3 columns, omitting the  $x_2$  and  $x_4$  columns and the right-hand side, so your row operations aren't acting on the actual equations. Build the full 4x5 augmented matrix in the order  $(x_1, x_2, x_3, x_4 \mid b)$ —it helps to first clear denominators by multiplying rows by 6, 30, 10, and 15 respectively—then create a leading 1 in the  $x_1$  column and eliminate below before moving to the next pivots.

Problem Solving Guide

AI Feedback

Submit Answer

# 예시: 벡터의 원상 구하기

Let  $V$  be the function subspace of  $\mathbb{R}^{\mathbb{R}}$  spanned by the linearly independent vector set  $M = \{\sin(1x), \sin(2x), \sin(3x)\}$ . Further, let  $f : V \rightarrow \mathbb{R}^3$  be a linear map. The images of the basis  $M$  of  $V$  under the map  $f$  are as follows:

$$\sin(1x) \mapsto \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \quad \sin(2x) \mapsto \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad \sin(3x) \mapsto \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

Determine a vector  $v \in V$  such that  $f(v) = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}$ .

How should I start here?

- ✦ 1) No work shown yet.
- 2) The missing step is expressing  $v$  as a linear combination of the basis and using linearity of  $f$  to set up equations.
- 3) Cause: conceptual gap on translating "find  $v$  with  $f(v)=\text{target}$ " into a solvable linear system.

Hint: Let  $v = a \sin(x) + b \sin(2x) + c \sin(3x)$ . Then by linearity,  
 $a(-2, 2, 0) + b(0, -1, 2) + c(2, -2, 2) = (-2, 5, -10)$ ,  
so solve the system

$[-2 \ 0 \ 2; \ 2 \ -1 \ -2; \ 0 \ 2 \ 2][a \ b \ c]^T = [-2 \ 5 \ -10]^T$  (or equivalently the three component equations).

$$a \cdot \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + c \cdot \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}$$

$$\left[ \begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 2 & -1 & -2 & 5 \\ 0 & 2 & 2 & -10 \end{array} \right]$$

$$R_2 \leftarrow R_2 + R_1$$

$$\left[ \begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 0 & -1 & 0 & 3 \\ 0 & 2 & 2 & -10 \end{array} \right]$$

$$R_3 \leftarrow R_3 + 2 \cdot R_2$$

$$\left[ \begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -4 \end{array} \right]$$

$$R_1 \leftarrow R_1 - R_3$$

$$\left[ \begin{array}{ccc|c} -2 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -4 \end{array} \right]$$

🌟 You've set up and row-reduced the system correctly but stopped just before extracting the coefficients. From your last matrix, interpret the equations:  $-2a = 2$ ,  $-b = 3$ , and  $2c = -4$ . Solve for  $a$ ,  $b$ ,  $c$ , then write  $v = a \sin(x) + b \sin(2x) + c \sin(3x)$  and verify that  $f(v)$  equals the target vector.

$$\left[ \begin{array}{ccc|c} -2 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -4 \end{array} \right]$$

$$R_1 \leftarrow \left(-\frac{1}{2}\right) \cdot R_1$$

$$R_2 \leftarrow (-1) \cdot R_2$$

$$R_3 \leftarrow \left(\frac{1}{2}\right) \cdot R_3$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & +2 \end{array} \right]$$

✨ Please recheck the final scaling step. From your row-reduced form,  $R_3 = [0 \ 0 \ 2 \mid -4]$ ; dividing by 2 should give  $R_3 = [0 \ 0 \ 1 \mid -2]$ , so the sign of  $c$  was flipped. With that fixed, read off  $a, b, c$  and write  $v = a \sin x + b \sin 2x + c \sin 3x$ , then verify by plugging into  $f$ .

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\Rightarrow a = -1, \quad b = -3, \quad c = -2$$

$$\Rightarrow v = -1 \sin(x) - 3 \sin(2x) - 2 \sin(3x)$$

✨ Good setup and clean row-reduction from the linear combination equation. As a quick self-check, plug your  $a$ ,  $b$ ,  $c$  into the three component equations to verify  $f(v) = (-2, 5, -10)$ ; if a posted answer disagrees, compare the basis order you used  $(\sin x, \sin 2x, \sin 3x)$  with theirs.

Verification :

$$(-1) \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+0-4 \\ -2+3+4 \\ 0-6-4 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}$$

✨ Nice work setting up the linear system from the images of the basis and using row reduction to find the coefficients. Your verification that the resulting linear combination reproduces  $(-2, 5, -10)^T$  shows the approach is consistent.

$$v = (-1)\sin(x) - 3\sin(2x) - 2\sin(3x)$$



Correct!

9번 만에 성공!

힌트는 14번 사용했네요.

이번 문제도 멋지게 풀었어요 🎉

📦 정답

$$v = -2\sin(3x) - 3\sin(2x) - 1\sin(1x)$$

# 과제: 정확한 해답 인식



$$x = -\sin(x) - 3\sin(2x) - 2\sin(3x)$$

$$\begin{aligned} a &= -1 \\ b &= -3 \\ c &= -2 \end{aligned}$$



Correct!

11번 만에 성공!

힌트는 14번 사용했네요.

이번 문제도 멋지게 풀었어요 🎉

📌 정답

$$v = -2\sin(3x) - 3\sin(2x) - 1\sin(1x)$$



Correct!

10번 만에 성공!

힌트는 14번 사용했네요.

이번 문제도 멋지게 풀었어요 🎉

📌 정답

$$v = -2\sin(3x) - 3\sin(2x) - 1\sin(1x)$$



# 과제: 강의 자료의 고려

$$\left[ \begin{array}{cccc|c} 1 & 3 & -4 & 3 & 9 \\ 3 & 9 & -2 & -11 & -3 \\ 4 & 12 & -6 & -8 & 6 \\ 2 & 6 & 2 & -14 & -12 \end{array} \right]$$

$$R_2 \leftarrow R_2 - 3R_1$$

$$R_3 \leftarrow R_3 - 4R_1$$

$$R_4 \leftarrow R_4 - 2R_1$$

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \textcircled{1} & 3 & -4 & 3 & 9 \\ 3 & 9 & -2 & -11 & -3 \\ 4 & 12 & -6 & -8 & 6 \\ 2 & 6 & 2 & -14 & -12 \end{array} \begin{array}{l} \cdot(-3) \quad \cdot(-4) \quad \cdot(-2) \\ + \\ + \\ + \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|c} & T & T & & \\ 1 & 3 & -4 & 3 & 9 \\ 0 & \textcircled{0} & 10 & -20 & -30 \\ 0 & 0 & 10 & -20 & -30 \\ 0 & 0 & 10 & -20 & -30 \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|c} x_1 & x_3 & x_2 & x_4 & \\ 1 & -4 & 3 & 3 & 9 \\ 0 & \textcircled{10} & 0 & -20 & -30 \\ 0 & 10 & 0 & -20 & -30 \\ 0 & 10 & 0 & -20 & -30 \end{array} \begin{array}{l} \\ \cdot \frac{1}{10} \\ \\ \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|c} 1 & -4 & 3 & 3 & 9 \\ 0 & \textcircled{1} & 0 & -2 & -3 \\ 0 & 10 & 0 & -20 & -30 \\ 0 & 10 & 0 & -20 & -30 \end{array} \begin{array}{l} + \\ \cdot(-10) \quad \cdot 4 \\ + \\ + \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|c} x_1 & x_3 & x_2 & x_4 & \\ 1 & 0 & 3 & -5 & -3 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

# 과제: 근거 제시 요구하기



Let  $V$  be the function subspace of  $\mathbb{R}^{\mathbb{R}}$  spanned by the linearly independent vector set  $M = \{\sin(1x), \sin(2x), \sin(3x)\}$ . Further, let  $f: V \rightarrow \mathbb{R}^3$  be a linear map. The images of the basis  $M$  of  $V$  under the map  $f$  are as follows:

$$\sin(1x) \mapsto \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \quad \sin(2x) \mapsto \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad \sin(3x) \mapsto \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

Determine a vector  $v \in V$  such that  $f(v) = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}$ .

This is a pre-image problem. With respect to the given basis images we are looking for a vector

$$v = \alpha \sin(1x) + \beta \sin(2x) + \gamma \sin(3x)$$

such that

$$f(v) = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}.$$

Since  $f$  is a linear map we may reason

$$\begin{aligned} f(v) &= f(\alpha \sin(1x) + \beta \sin(2x) + \gamma \sin(3x)) \\ &= \alpha f(\sin(1x)) + \beta f(\sin(2x)) + \gamma f(\sin(3x)) \\ &= \alpha \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix} \end{aligned}$$

Reading the final equality component-wise, we obtain a linear system that can be solved using elimination:

$$\begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 2 & -1 & -2 & 5 \\ 0 & 2 & 2 & -10 \end{array}$$



- 초기 개발 단계임에도 불구하고 학생들에게 큰 도움이 됨
- 자기 점검을 돕고 좌절감을 줄이는 긍정적 효과
- 관찰된 단점들은 치명적이지 않은 것으로 평가됨
- 지속적인 개선을 위해 연구 결과 활용