Ostfalia University of Applied Sciences

오스트팔리아 대학교의 한림 수학 AI 튜터 사례 연구

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자기 소개

- **약력**
  - 2004: 박사 학위 취득
  - 2009: 교수자격 취득 (독일 교육 시스템의 특성)
  - 2005~2010: 소프트웨어 아키텍트로 활동
  - 2010 이후: 오스트팔리아 대학교 재직
- **주요 연구 분야:** 그래프 이론
- **주요 강의 주제:** IT 보안, 데이터베이스, 운용 과학, 비즈니스 프로세스, 수학

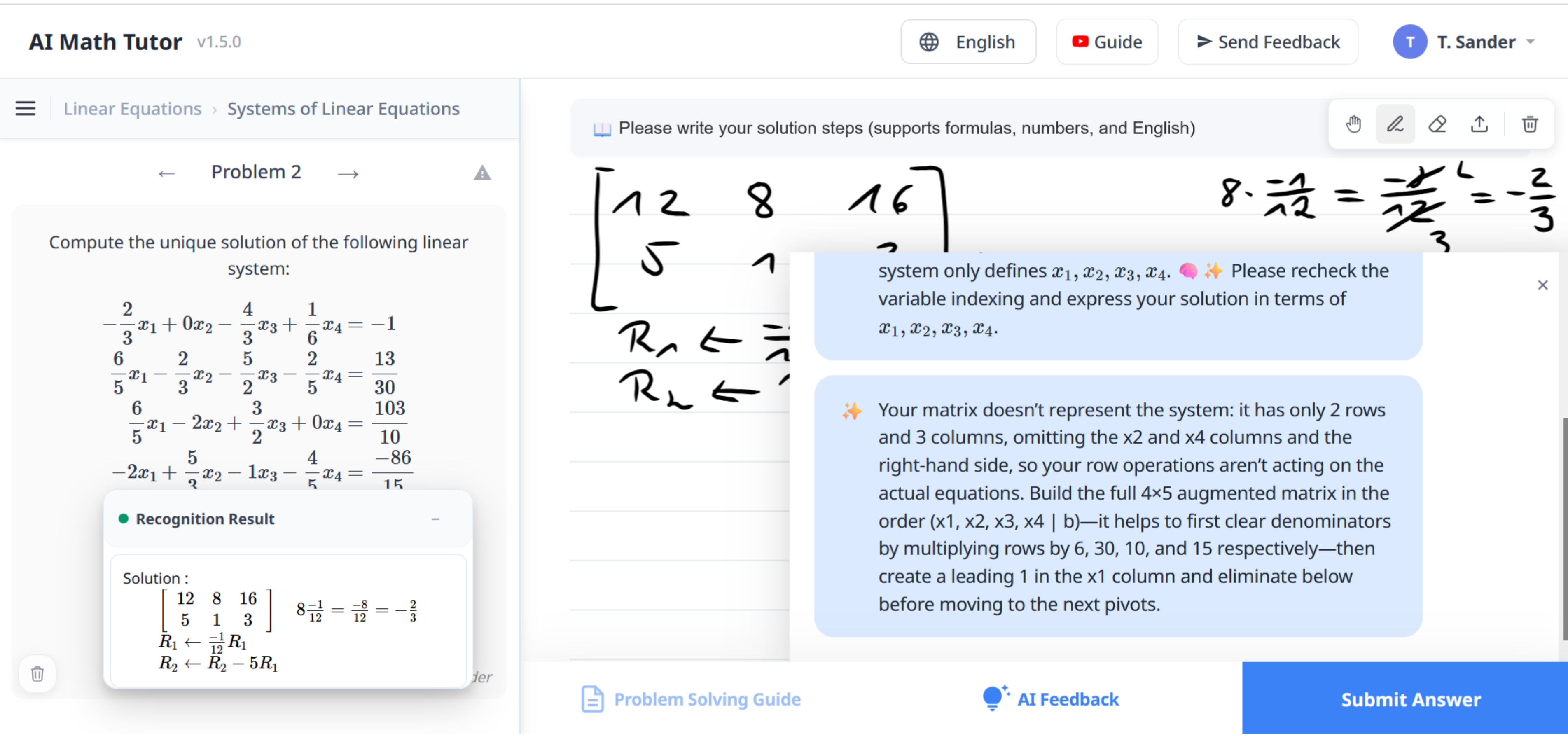
사례 연구 배경

- 한림대학교 '수학 AI 튜터' 연구 프로젝트
- 초기 개발 단계
- 활발한 개발 및 주간 단위 업데이트
- 오스트팔리아 대학의 역할: 초기 사용 경험 축적 및 피드백 제공
- 목표 대상: 선형대수학을 수강하는 학부생

주요 연구 질문

- 풀이 과정에 대한 적절한 힌트 제공 여부
- 힌트의 상세 수준: 너무 많거나 적지 않은가?
  – 이전 힌트와 다음 힌트의 일관성
  – 계산 오류의 정확한 탐지 및 지적
- 독일과 한국 간의 문화적, 교육적 차이

수학 AI 튜터: 인터페이스 및 워크플로우



예시: 벡터의 원상 구하기

Let V be the function subspace of $\mathbb{R}^{\mathbb{R}}$ spanned by the linearly independent vector set $M = \{\sin(1x), \sin(2x), \sin(3x)\}$. Further, let $f: V \to \mathbb{R}^3$ be a linear map. The images of the basis M of V under the map f are as follows:

$$\sin(1x) \mapsto \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \quad \sin(2x) \mapsto \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad \sin(3x) \mapsto \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

Determine a vector $v \in V$ such that $f(v) = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}$.

How should I start here?

✨ 1) No work shown yet.
2) The missing step is expressing v as a linear combination of the basis and using linearity of f to set up equations.
3) Cause: conceptual gap on translating “find v with f(v)=target” into a solvable linear system.

Hint: Let v = a sin(x) + b sin(2x) + c sin(3x). Then by linearity,
a(-2,2,0) + b(0,-1,2) + c(2,-2,2) = (-2,5,-10),
so solve the system
[-2 0 2; 2 -1 -2; 0 2 2][a b c]^T = [-2 5 -10]^T (or equivalently the three component equations).

$$a \cdot \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + c \cdot \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 2 & -1 & -2 & 5 \\ 0 & 2 & 2 & -10 \end{array}\right]$$

$R_2 \leftarrow R_2 + R_1$

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 0 & -1 & 0 & 3 \\ 0 & 2 & 2 & -10 \end{array}\right]$$

$R_3 \leftarrow R_3 + 2 \cdot R_2$

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_3$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

✨ You've set up and row-reduced the system correctly but stopped just before extracting the coefficients. From your last matrix, interpret the equations: -2a = 2, -b = 3, and 2c = -4. Solve for a, b, c, then write v = a sin(x) + b sin(2x) + c sin(3x) and verify that f(v) equals the target vector.

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

$$R_1 \leftarrow \left(-\tfrac{1}{2}\right) \cdot R_1$$
$$R_2 \leftarrow (-1) \cdot R_2$$
$$R_3 \leftarrow \left(\tfrac{1}{2}\right) \cdot R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & +2 \end{array}\right]$$

✨ Please recheck the final scaling step. From your row-reduced form, $R_3 = [\,0\;0\;2 \mid -4\,]$; dividing by 2 should give $R_3 = [\,0\;0\;1 \mid -2\,]$, so the sign of c was flipped. With that fixed, read off a, b, c and write $v = a\sin x + b\sin 2x + c\sin 3x$, then verify by plugging into f.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\Rightarrow a = -1 , \quad b = -3 , \quad c = -2$$

$$\Rightarrow v = -1 \sin(x) - 3 \sin(2x) - 2 \sin(3x)$$

✨ Good setup and clean row-reduction from the linear combination equation. As a quick self-check, plug your a, b, c into the three component equations to verify f(v) = (-2, 5, -10); if a posted answer disagrees, compare the basis order you used (sin x, sin 2x, sin 3x) with theirs.

Verification :

$$(-1)\begin{pmatrix}-2\\2\\0\end{pmatrix} - 3\begin{pmatrix}0\\-1\\2\end{pmatrix} - 2\begin{pmatrix}2\\-2\\2\end{pmatrix} = \begin{pmatrix}2+0-4\\-2+3+4\\0-6-4\end{pmatrix} = \begin{pmatrix}-2\\5\\-10\end{pmatrix}$$

✨ Nice work setting up the linear system from the images of the basis and using row reduction to find the coefficients. Your verification that the resulting linear combination reproduces $(-2, 5, -10)^T$ shows the approach is consistent.

$$v = (-1)\sin(x) - 3\sin(2x) - 2\sin(3x)$$

×

🎉

Correct!

9번 만에 성공!

힌트는 **14번** 사용했네요.

이번 문제도 멋지게 풀었어요 👏

정답

$v = -2\sin(3x) - 3\sin(2x) - 1\sin(1x)$

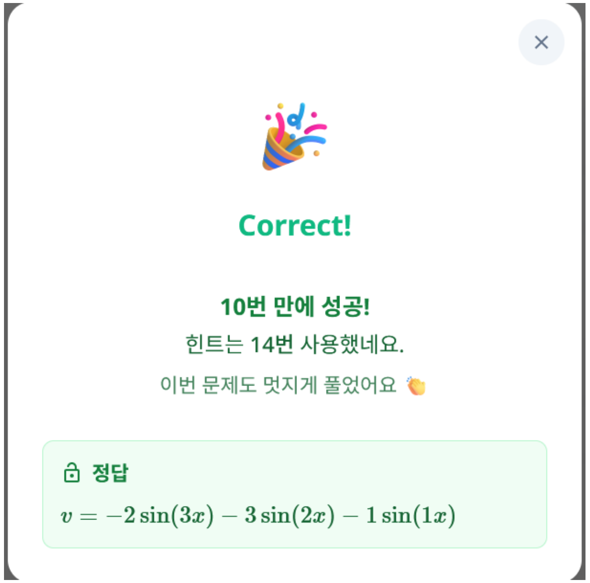
과제: 정확한 해답 인식

$$x = -\sin(x) - 3\sin(2x) - 2\sin(3x)$$

$a = -1$

$b = -3$

$c = -2$



과제: 강의 자료의 고려

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 3 & 9 \\ 3 & 9 & -2 & -11 & -3 \\ 4 & 12 & -6 & -8 & 6 \\ 2 & 6 & 2 & -14 & -12 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$

$R_3 \leftarrow R_3 - 4R_1$

$R_4 \leftarrow R_4 - 2R_1$

$$\begin{array}{cccc|cccc} x_1 & x_2 & x_3 & x_4 & & & & \\ 1 & 3 & -4 & 3 & 9 & \cdot(-3) & \cdot(-4) & \cdot(-2) \\ 3 & 9 & -2 & -11 & -3 & + & & \\ 4 & 12 & -6 & -8 & 6 & & + & \\ 2 & 6 & 2 & -14 & -12 & & & + \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|c} & T & T & & \\ 1 & 3 & -4 & 3 & 9 \\ 0 & 0 & 10 & -20 & -30 \\ 0 & 0 & 10 & -20 & -30 \\ 0 & 0 & 10 & -20 & -30 \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|cc} x_1 & x_3 & x_2 & x_4 & & \\ 1 & -4 & 3 & 3 & 9 & \\ 0 & 10 & 0 & -20 & -30 & \cdot\frac{1}{10} \\ 0 & 10 & 0 & -20 & -30 & \\ 0 & 10 & 0 & -20 & -30 & \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|ccc} 1 & -4 & 3 & 3 & 9 & & + \\ 0 & 1 & 0 & -2 & -3 & \cdot(-10) & \cdot 4 \\ 0 & 10 & 0 & -20 & -30 & + & \\ 0 & 10 & 0 & -20 & -30 & + & \end{array}$$

$$\Leftrightarrow \begin{array}{cccc|c} x_1 & x_3 & x_2 & x_4 & \\ 1 & 0 & 3 & -5 & -3 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

과제: 근거 제시 요구하기

Let V be the function subspace of $\mathbb{R}^{\mathbb{R}}$ spanned by the linearly independent vector set $M = \{\sin(1x), \sin(2x), \sin(3x)\}$. Further, let $f: V \to \mathbb{R}^3$ be a linear map. The images of the basis M of V under the map f are as follows:

$$\sin(1x) \mapsto \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \quad \sin(2x) \mapsto \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad \sin(3x) \mapsto \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

Determine a vector $v \in V$ such that $f(v) = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}$.

This is a pre-image problem. With respect to the given basis images we are looking for a vector

$$v = \alpha \sin(1x) + \beta \sin(2x) + \gamma \sin(3x)$$

such that

$$f(v) = \begin{pmatrix} -2 \\ 5 \\ -10 \end{pmatrix}.$$

Since f is a linear map we may reason

$$\begin{aligned} f(v) &= f\big(\alpha \sin(1x) + \beta \sin(2x) + \gamma \sin(3x)\big) \\ &= \alpha f(\sin(1x)) + \beta f(\sin(2x)) + \gamma f(\sin(3x)) \\ &= \alpha \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} -2 \\ 5 \\ 10 \end{pmatrix} \end{aligned}$$

Reading the final equality component-wise, we obtain a linear system that can be solved using elimination:

$$\left(\begin{array}{ccc|c} -2 & 0 & 2 & -2 \\ 2 & -1 & -2 & 5 \\ 0 & 2 & 2 & 10 \end{array}\right)$$

요약

- 초기 개발 단계임에도 불구하고 학생들에게 큰 도움이 됨
- 자기 점검을 돕고 좌절감을 줄이는 긍정적 효과
- 관찰된 단점들은 치명적이지 않은 것으로 평가됨
- 지속적인 개선을 위해 연구 결과 활용